

# Markov Chain Hidden behind Power Laws Mechanism of Self-organized Criticality

De Tao, Mao\*

*Electrical and Computer Engineering Department, University of British Columbia.*

Yisheng, Zhong†

*Automation Department, Tsinghua University.*

(Dated: October 5, 2009)

To describe and analyze the dynamics of Self-Organized Criticality (SOC) systems, a four-state continuous-time Markov model is proposed in this paper. Different to computer simulation or numeric experimental approaches commonly employed for explaining the power law in SOC, in this paper, based on this Markov model, using E.T.Jayness' Maximum Entropy method, we have derived a mathematical proof on the power law distribution for the size of these events. Both this Makov model and the mathematical proof on power law present a new angle on the universality of power law distributions, they also show that the scale free property exists not necessary only in SOC system, but in a class of dynamical systems which can be modelled by the proposed Markov model.

PACS numbers: **05.65+b,02.50.Ga**

## I. INTRODUCTION

As a distinctive signature, power law exists in many real-world systems, such as forest fire [1] [2] [3], epidemic [4][5], avalanche, social wealth distribution [6], blackout of power systems [7][8], and earthquake [9] etc. It is also given various names in these above areas, for instance, "Pareto distribution", "Zipf's law", "scale free distribution", and "fat tails" etc [10].

Since the famous work of BTW (Bak-Tang-Wiesenfeld) [11][12], which was mainly on proving that Self-Organized Criticality (SOC) was a potential underlying mechanism of any system with the fingerprint of power law, there are increasing research efforts on studying the issue of ubiquitous existence of this scale free property in nature. However, most results are obtained via comprehensive experiments of computer simulation, but the problem why SOC systems and many other systems are ruled by power law, is still mathematically unsolved.

In this study, a Continuous-time Hidden Markov model is proposed, with which a rigorous proof on the power law distribution is derived analytically. This study also provides insights into this novel combination of Continuous-time Markov model and the Maximum Entropy Principle [13], leading to a better understanding on the dynamics of these systems with power law signature.

The rest of this paper is organized as follows, Section II briefly introduces some background conceptions, and also a hidden Markov model is constructed to describe the evolution process of SOC. These descriptive equations in different states are introduced in Section III. The proof on power law in SOC systems is given in Section

IV, simulation results are displayed in Section V. Finally, Section VI will give some discussions, conclusions and possible future work of this study.

## II. BACKGROUND CONCEPTIONS AND THE PROPOSED HIDDEN MARKOV MODEL

This work is closely related to Self-Organized criticality and Continuous-time Hidden Markov Chain theory. A brief introduction to the basic conceptions and the proposed model are given here.

### A. Self-organized Criticality

Self-organized Criticality is a universal property of systems far from equilibrium, and it is initially proposed by BTW (Bak-Tang-Wiesenfeld) in [11][12][14][15]. From then on it has been widely applied to describe the macroscopic dynamical behavior of an open system near its critical point, where the system displays a scale invariance feature. But different to phase transition process in classical physics, at critical point, it is unnecessary for the SOC system to tune its parameters to a set of concrete values. And BTW's model demonstrates that the observed complexity which is emerged in a robust manner, does not depend on the finely-tuned details of the system, parameters of the model could be changed widely without affecting the emergence of critical behavior.

### B. Continuous-time Hidden Markov Model

When observing SOC phenomena in nature, such as forest fire, earthquake, and epidemics etc, the most obvious characteristics is their unexpected emergence. Hence

---

\*Electronic address: [detaom@ece.ubc.ca](mailto:detaom@ece.ubc.ca)

†Electronic address: [zys-dau@mail.tsinghua.edu.cn](mailto:zys-dau@mail.tsinghua.edu.cn)

intuitively, their evolution process can be roughly modelled into two states: hibernating state and active state. However, considering the underlying mechanism for the stochastic character of SOC, an active state is not sufficient to describe the complexity of this dynamical property. Thus, the active state is divided into three sub-states: self-sustaining state, declining state and autocatalytic state. An intermediate state—self-sustaining state is introduced here, because for a continuous-time model, the transition interval between declining state and autocatalytic state cannot be infinitesimal.

Suppose for this four-state state machine, the transition rate between state  $i$  and state  $j$  is  $P(E)_{ij}$ , since usually this transition is triggered by events or induced by a certain type of conditions, thus this event-triggered or condition-induced transition probability can be written as,

$$P(E) = P(E_{ex}) + P(E_{en}|C)P(C) - P(E_{ex} \cap E_{en} \cap C)^{[1]}$$

Where  $E_{ex}$  denotes an exogenous event that will trigger a transition from one state to another,  $E_{en}$  denotes an endogenous event exists only under a certain condition  $C$ , and  $P(E_{ex})$ ,  $P(E_{en})$ ,  $P(C)$  are the occurrence probabilities of event  $E_{ex}$ , event  $E_{en}$ , condition  $C$ , respectively.

Assuming these event-triggered and/or condition-induced transition probabilities  $P(E)_{ij}$  are all statistically stable. Mathematically, these processes can be modelled as a Continuous-time Markov chain. This four-state Markov model can be drawn in Fig. 1. Its transition rates  $q_{ij}$  are typically described as the  $ij$ -th elements of the Infinitesimal generator matrix  $Q$ . In a infinitesimal time interval  $\Delta t$ ,  $Q$  can be expressed as

$$Q = \begin{pmatrix} -q_H & q_{HS} & 0 & 0 \\ 0 & -q_S & q_{SD} & q_{SA} \\ q_{DH} & q_{DS} & -q_D & 0 \\ 0 & q_{AS} & 0 & -q_A \end{pmatrix} \quad (1)$$

where  $q_{ij}$  is the transition rate between state  $i$  and state  $j$ ,  $q_i$  is the self-transition rate of state  $i$ .

When time approaches infinity, the stationary distribution of the markov states  $\pi = (\pi_H, \pi_S, \pi_D, \pi_A)$  can be calculated by

$$\pi = e \cdot (Q + E)^{-1} = (\pi_H, \pi_S, \pi_D, \pi_A) \quad (2)$$

Where

$$E = \begin{pmatrix} 1 & \dots & 1 \\ \dots & & \\ 1 & \dots & 1 \end{pmatrix}_{4 \times 4},$$

$$e = [1, 1, 1, 1], \pi_H + \pi_S + \pi_D + \pi_A = 1.$$

[1] In  $P(E)$ , we only count independent exogenous or endogenous events, these events that belong to an event cascade caused by  $E_{ex}$  and/or  $E_{en}$  are not considered.

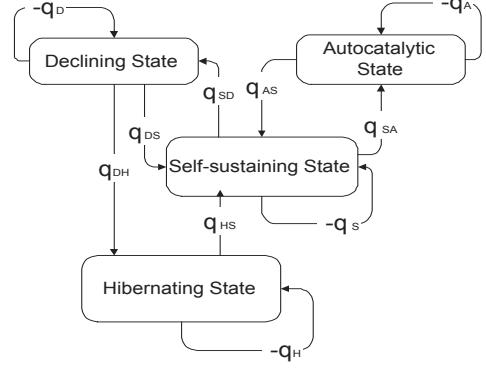


FIG. 1: Four States Continuous-Time Hidden Markov Model for evolution processes in Self-organized Criticality.

and  $\pi_H, \pi_S, \pi_D, \pi_A$  denotes the stationary distribution of Hibernating state, Self-sustaining state, Declining state, and Autocatalytic state, repectively.

### III. THE EVOLUTION EQUATIONS IN DIFFERENT STATES

For a SOC system far from equilibrium, when it is approaching its critical point, instead of entering a state of phase transition entirely, parts of its unit population will evolve into a meta-stable state.

To demonstrate the evolution process of SOC systems, we can analyze its activities in autocatalytic state firstly. In an autocatalytic state, as most of these clustered meta-stable units are in critical state, a small event can trigger a large disaster that propagates quickly as well as greatly changes the stable-degree distribution of these units.

For an ongoing event in autocatalytic state, let  $\xi = \frac{d\vec{n}}{dt}$  be the unfolding speed at the normal direction  $\vec{n}$  of this event, via a first-ordered linear approximation,  $\xi = \frac{d\vec{n}}{dt}$  can be determined in proportion to the divergence  $(\nabla \cdot \Phi)$  of the energy flux  $\Phi$  dissipating across its boundary. For instance, in BTW's sandpile model for SOC, during an event, these triggered meta-stable units will release their energy to these peripheral units, among which these meta-stable ones will be triggered to release their energy, and these stable one will absorb the energy and become unstable. The equations to describe the above processes is derived as,

$$\begin{aligned} \xi &= \frac{d\vec{n}}{dt} = k_1 \nabla \cdot \Phi \\ \nabla \cdot \Phi &= \frac{\oint d\Phi}{\oint dl} \\ \oint dl \cdot d\vec{n} &= dA \end{aligned} \quad (3)$$

As the total energy flux  $\Phi$  produced by the event is in

proportion to its current size  $A$ , thus

$$\oint_A d\Phi = k_2 \iint d\sigma = k_2 \cdot A \quad (4)$$

Where  $k_1$  and  $k_2$  denote these linear coefficients,  $d\sigma$  is the differential of area  $A$ . The final expression of the above equations can be converted to:

$$\frac{dA}{dt} = k_1 \cdot k_2 A = \dot{\mathbf{r}}^+ A \quad (\dot{\mathbf{r}}^+ > 0) \quad (5)$$

As observed in the above equation, the coefficient  $\dot{\mathbf{r}}^+$  stands for the intensity of positive-feedback effect in autocatalytic state, which is determined by the dependency range or correlated degree among the different units in the system.

Accompanying an ongoing event, a similar analysis process can also be applied on its declining states, for an event cannot trigger these meta-stable units to make them release energy all the time, at micro time scale, these units at the event's boundary may absorb these released energy, making these meta-stable units become stable and the possible large event will shrink to a small size, this can be seen as an reversible process of the autocatalytic state, similar to the above analysis, we can get an equation to describe the declining state,

$$\frac{dA}{dt} = \dot{\mathbf{r}}^- A \quad (\dot{\mathbf{r}}^- < 0) \quad (6)$$

Similarly, the coefficient  $\dot{\mathbf{r}}^-$  stands for the intensity of negative-feedback effect in declining state, which is also determined by the dependency range or correlated degree among the different units in the system.

As in hibernating state there is no event, while in self-sustaining state, there is an ongoing event in the system. It is obvious that in self-sustaining state, the event's size keeps invariant, therefore the evolution time in this state can be omitted *in our model*, since only the declining state and the autocatalytic state are essential in determining the final size of the event. An evolution equation for the size of the event can be proposed as follows:

$$\frac{dX(t)}{dt} = [\dot{\mathbf{r}}^+ \cdot \delta(S_t - S_A) + \dot{\mathbf{r}}^- \cdot \delta(S_t - S_D)] X(t) \quad t = 0 : X = X_0. \quad (7)$$

In Eqn.(7),  $X(t)$  is the size of the event at time  $t$ ,  $\delta$  denotes Dirac function,  $S_t$  means the system's state at time  $t$ ,  $S_A$  ( $S_D$ ) denotes the system's autocatalytic (declining) state at time  $t$ ;  $\dot{\mathbf{r}}^+$  ( $\dot{\mathbf{r}}^-$ ) corresponds to the expanding (shrinking) speed of meta-stable units in an autocatalytic (declining) state.

According to Eqn.(7) above, an event's final size can be expressed as

$$X(t) = X_0 \cdot e^{\dot{\mathbf{r}}^+ \sum_i \mathbf{t}_i^A + \dot{\mathbf{r}}^- \sum_j \mathbf{t}_j^D} \quad (8)$$

Where  $\sum_i \mathbf{t}_i^A$  ( $\sum_j \mathbf{t}_j^D$ ) represents the accumulated duration that the event has stayed at the autocatalytic (declining) state.  $i$  ( $j$ ) represents that how many times that the event has visited the autocatalytic (declining) state.

#### IV. SOC AS A STATIONARY PROCESS IN THE VIEW OF THE LAW OF LARGE NUMBERS

##### A. Stable Interaction Hypothesis

In SOC process, such as forest-fire model and BTW's sandpile model, let  $\mathcal{S}_s(t)$ ,  $\mathcal{S}_a(t)$  and  $\mathcal{S}_d(t)$  denote as the "negative entropy"<sup>[2]</sup> stored in system, absorbed from the surrounding environment, and dissipated into the environment, respectively. Via the laws of conservation, there exists the following relation,

$$\mathcal{S}_d(t) = \mathcal{S}_a(t) - \mathcal{S}_s(t) \quad (9)$$

Which means in a short term,  $\mathcal{S}_d(t)$  is temporally determined by  $\mathcal{S}_a(t)$  and  $\mathcal{S}_s(t)$ .

However, in the long term, given the system's environments keep stable, i.e.,  $\mathcal{S}_a(t)$  and  $\mathcal{S}_d(t)$  are all stationary processes, as for any SOC system under study, the negative entropy stored in it  $\mathcal{S}_s(t)$  can be deemed as a finite quantity. Thus when time goes infinity,

$$\lim_{(t-t_0) \rightarrow \infty} \frac{\int_{t_0}^t \mathcal{S}_a(\tau) d\tau}{t - t_0} = \lim_{(t-t_0) \rightarrow \infty} \frac{\int_{t_0}^t \mathcal{S}_d(\tau) d\tau}{t - t_0} \quad (10)$$

which means in long term, the entropy absorbed from the environment and dissipated into it can be deemed as equal.

For these processes with property of statistical stationary, determined by Eqn.(9) and Eqn.(10), the average quantity of negative entropy exchanged between the system and its environment  $\Delta S_{ex}$  keeps at an invariant level. Given any length-fixed time window  $[t_i, t_i + \Delta T]$ , the number of events  $n_i$  within it may fluctuate from time to time. But in a long term, for  $n$  times of observations, according to the law of large numbers, the average number of observable events  $\bar{n} = \frac{\sum_i n_i}{n}$  within this time window  $\Delta T$  can be regarded invariant, therefore the average time interval between events  $T_0 = \frac{\Delta T}{\bar{n}}$  should also be an invariant quantity.

As shown in above sections, SOC process has been modelled by a continuous-time Hidden Markov Chain. During an event, the average time in its autocatalytic state (declining state) is  $\pi_A T_0$  ( $\pi_D T_0$ ), and from the properties of continuous-time Hidden Markov model, obviously,  $T_0$ ,  $\pi_A$  and  $\pi_D$  are all invariant quantities.

---

[2] The "negative entropy" here can be defined in a general way, including mass  $M$ , entropy  $S$ , information  $I$ , energy  $E$  and other combinational forms of them.

## B. Proof of Power Law Distribution in SOC

Suppose  $x_m$  denote the minimum size of an observable event in the SOC system under study, for instance, in BTW's sandpile model,  $x_m$  might represent a sand grain, and in forest fire model,  $x_m$  might represent a tree. Assuming that all the observable events in the system have the same initial size  $x_m$ , by the solution of Eqn.(8), it is obvious that only when  $(\bar{\mathbf{r}}\pi_A + \bar{\mathbf{r}}\pi_D) \geq 0$ , the event is observable.

During its evolution process, assuming  $N$  events with different size  $X_i$  ( $X_i \geq x_m, i = 1, \dots, N$ ) can be observed in the SOC system. Suppose they are ruled by a distribution  $f(x)$ , to histogram them into a double logarithm coordinates via their sizes, these events can be quantized into  $m$  different sets or intervals, for instance, in set  $l$  ( $l \in \{1, \dots, m\}$ ), there are  $N_l$  ( $\sum_{l=1}^m N_l = N$ ) events, and in this set, the size difference between event  $X_l^j$  and event  $X_l^k$  should be less than  $\delta_{X_l}$ . Here  $\delta_{X_l}$  is the threshold value for set  $l$ , ensuring in it  $\prod_{j=1}^{N_l} (X_l^j)$  is almost equal to  $(X_l)^{N_l}$ .

Let define quantity  $L$  as

$$L \stackrel{\text{def}}{=} \prod_{i=1}^N X_i = \prod_{l=1}^m (X_l)^{N_l} \quad (11)$$

Based on Eqn.(8),  $L$  can be expressed as,

$$L = x_m^N e^{\sum (\bar{\mathbf{r}}\mathbf{t}_{1,i}^A + \bar{\mathbf{r}}\mathbf{t}_{1,i}^D) N_l} \quad (12)$$

In Eqn.(12),  $\mathbf{t}_{l,i}^A$  ( $\mathbf{t}_{l,i}^D$ ) means the duration that the system stays at the autocatalytic (declining) state in the event  $i$  of set  $l$ .

Let define quantity  $\psi$  as

$$\psi \stackrel{\text{def}}{=} \frac{\ln L}{N} = \ln(x_m e^{\sum (\bar{\mathbf{r}}\mathbf{t}_{1,i}^A + \bar{\mathbf{r}}\mathbf{t}_{1,i}^D) \frac{N_l}{N}}) \quad (13)$$

It can also be expressed as

$$\psi = \frac{\ln(\prod (X_l)^{N_l})}{N} = \sum \frac{N_l}{N} \ln(X_l) \quad (14)$$

As

$$\lim_{\delta_{X_l} \rightarrow 0} \frac{N_l}{N} \rightarrow f(X_l) \quad (15)$$

as  $f(x)$  denotes as the events' probability density function. Eqn. (14) can be expressed as

$$\psi = \int_{x_m}^{\infty} f(x) \ln x \, dx \quad (16)$$

therefore  $\psi$  is equal to the expectation of  $\ln x$ , as the stationary distribution of this continuous-time Markov chain is  $\pi = (\pi_H, \pi_S, \pi_D, \pi_A)$ , and also because

$$\psi = E\{\ln x\} = \ln(x_m) + (\bar{\mathbf{r}} \cdot \pi_A + \bar{\mathbf{r}} \cdot \pi_D) \cdot T_0 \quad (17)$$

as this SOC process has been assumed as a statistical stationary process,  $x_m$ ,  $\bar{\mathbf{r}}$ ,  $\pi_D$ ,  $\pi_A$  and  $T_0$  are all constants,  $\psi$  can be deemed as an invariant quantity.

The entropy of the system can be expressed as

$$J(f) = - \int_{x_m}^{\infty} f(x) \ln f(x) \, dx \quad (18)$$

according to Eqn.(17) and normalization condition, via lagrangian multiplier method, the above equation can be converted in the following form:

$$\begin{aligned} J(f) = & - \int_{x_m}^{\infty} f(x) \ln f(x) \, dx + \lambda_1 \left( \int_{x_m}^{\infty} f(x) \, dx - 1 \right) \\ & + \lambda_2 \left( \int_{x_m}^{\infty} f(x) \ln x \, dx - \psi \right) \end{aligned} \quad (19)$$

By E.T.Jaynes' Maximum Entropy Principle [16], we can get

$$f(x) = e^{\lambda_1 x^{\lambda_2}} \quad (20)$$

let  $\alpha = -(\lambda_2 + 1)$ , we can get

$$f(x) = \alpha x_m^{\alpha} x^{-\alpha-1} \quad (\alpha > 0) \quad (21)$$

As

$$E\{\ln x\} = \int_{x_m}^{\infty} \alpha x_m^{\alpha} x^{-\alpha-1} \ln x \, dx = \ln x_m + \frac{1}{\alpha} \quad (22)$$

from Eqn.(17) and Eqn.(22),

$$\alpha = [(\bar{\mathbf{r}}\pi_A + \bar{\mathbf{r}}\pi_D)T_0]^{-1} \quad (23)$$

## V. SIMULATION RESULTS

Since the average periodicity or interval  $T_0$  between events is a constant quantity, and the distribution for the holding times of this four-state Markov model  $\{\pi_A, \pi_D, \pi_H, \pi_S\}$  is stationary, using the discrete time method introduced in [17], we get different power law distribution by simulation at different parameters  $\bar{\mathbf{r}}$ , as shown in Fig. 2.

## VI. CONCLUSION AND FUTURE WORK

The SOC system under study and the environments are all supposed to be statistically stationary, this can be correct for almost all cases in real-world and artificial systems. Though observed in a short term, these parameters, like transition rates, are usually not as static as we expected, which means flux of energy, mass, information and their combinational forms are not constant quantities, the frequency of event will fluctuate in this short

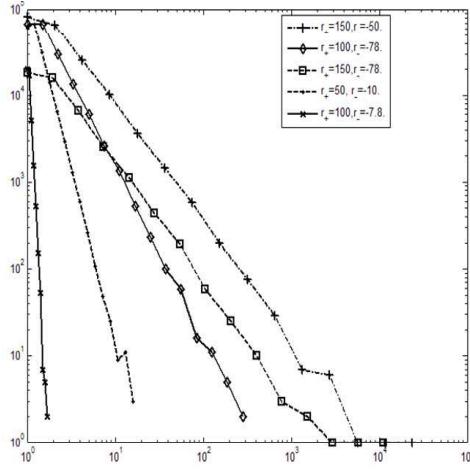


FIG. 2: Power Law distributions with different simulation parameters. where  $r_{\pm}$  is equal to  $\bar{r}$  noted in the above sections.

term.

This continuous-time Hidden Markov Model can give us a framework to describe the dynamical properties not only for these SOC systems, but for these regular systems different to SOC, yet owning positive and/or negative feed-back states during their evolution processes. The ubiquity of power laws is mainly because the ubiquity of these systems that can be described by this Markov model.

One potential direction based on this model is to unify these stochastic processes with power law fingerprint, such as Geometric Brownian motion, Discrete multiplicative process, Homogeneous birth-and-death process, and Galton-Walton branching process etc in [18], Forrest-fire model in [19], and Yule process in [20] etc.

As human behavior can affect the bidirectional flux between the systems and its environments, there is possibility on controlling the events frequency and their size distribution, so as to reduce the probability of large scale events and prevent the collapse of the whole systems. Some applications and academic work based on Highly Optimized Tolerance (HOT) have been done in [2][3] [21]

[22][23] etc, but as our model has different features from HOT, a lot of research studies are still very necessary at this direction.

## APPENDIX A: LIFE TIME OF AN EVENT AND ITS MULTI-TIME SCALE PROPERTY

When the state of a SOC system is in its hibernating state, there is no event in the system, otherwise when an event happens, by passing through the self-sustaining state, the system will escape its hibernating state into any one of these three active states – self-sustaining, declining and autocatalytic states. If the event keeps on, the system state will jump among the active states

FIG. 3: Continuous-time Markov chain demonstrated along time axis  $t$ .

stochastically. Therefore in this Markov model, the lifetime of an event is the duration since its last escape from the hibernating state until the next return to it.

The multi-time scale property of this continuous-time Markov Chain model is displayed in the following Fig.3,where A(D,S,H) stands for the four states of the Markov model respectively. The framed section is zoomed out to show the detail information of this Multi-time scale model.

- 
- [1] W.J.Reed and B.D.Hughes, Can. J. For. Res. **34**, 2306 (2004).
  - [2] J.M.Carlson and J. Doyle, Proc. Natl. Acad. Sci.,USA **99**, 2538 (2002,Suppl.1.).
  - [3] J. Doyle and J. M. Carlson, Physical Review Letters **84** (2000).
  - [4] R. Pastor-Satorras and A. Vespignani., PHYS.REV.LETT. **86**, pp.3200 (2001).
  - [5] Z. Deszo and A. L. Barabasi, arXivcond-mat/0107430 **2** (2002).
  - [6] J. K. N. P. Burda.Z, Johnston.D and Zahed.I, arXiv cond-mat/0101068 **1** (2001).
  - [7] I. B.A.Carreras, D.E.Newman and A.B.Poole, in *In 34th Hawaii Internaltional Conference on System Sciences, IEEE Computer Society.* (2001).
  - [8] J. J.Chen and M.Parashar, in *In 34th Hawaii Internaltional Conference on System Sciences, IEEE Computer Society( pp.705-709.)* (2001).
  - [9] C.G.Sammis and D.Sornette., PNAS **99**, 2501 (2002).
  - [10] Mark.E.J.Newman, Nature **405**, 412 (May 25th 2000).
  - [11] C. T. P.Bak and K.Wiesenfeld, Phys. Rev.A **38**, 364 (July 1, 1988).
  - [12] C. T. Per.Bak and K. Wiesenfeld, Phys.Rev. **38**, 364 (1988).
  - [13] E.T.Jayness, *Probability Theory — The Logic of Science* (Cambridge University Press, 2003).

- [14] Per.Bak and M. Paczuski, PNAS. USA **92**, 6689 (July 1995).
- [15] Per.Bak, *How Nature Works:The Science of Self-organized Criticality* (Oxford University Press, 1997).
- [16] E.T.Jaynes, Phys. Rev. **106**. NO.**4**, 620 (May 15, 1957).
- [17] D. L. S. R. Hordij, Arie; Iglehart, Adv.Appl.Prob. **8**, 772 (1976).
- [18] R. W.J and B.D.Hughes, Phys. Rev. E **66** (2002).
- [19] F. B.Drossel, Phys. Rev. Letters **69**, No **11**,, 1629 (1992).
- [20] M.E.J.Newman, arXiv:cond-mat/0412004 **2**,No.**9** (Jan 2005).
- [21] T. Zhou and J.M.Carlson, Phys. Rev. E. **62**, No. **3**, 3197 (Sep 2000).
- [22] J. Tong Zhou and J. Doyle, PNAS **99**, 2049 (Feb 192002).
- [23] J.M.Carlson and J. Doyle, Phys. Rev. E. **60**,No **2**., 1412 (Aug 1999).